

Statistics: 2.2 The Wilcoxon signed rank sum test

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1 Introduction

The Wilcoxon signed rank sum test is another example of a non-parametric or distribution free test (see 2.1 The Sign Test). As for the sign test, the Wilcoxon signed rank sum test is used to test the null hypothesis that the median of a distribution is equal to some value. It can be used a) in place of a one-sample t-test b) in place of a paired t-test or c) for ordered categorical data where a numerical scale is inappropriate but where it is possible to rank the observations.

2 Carrying out the Wilcoxon signed rank sum test

Case 1: Paired data

1. State the null hypothesis - in this case it is that the median difference, M , is equal to zero.
2. Calculate each paired difference, $d_i = x_i - y_i$, where x_i, y_i are the pairs of observations.
3. Rank the d_i s, ignoring the signs (i.e. assign rank 1 to the smallest $|d_i|$, rank 2 to the next etc.)
4. Label each rank with its sign, according to the sign of d_i .
5. Calculate W^+ , the sum of the ranks of the positive d_i s, and W^- , the sum of the ranks of the negative d_i s. (As a check the total, $W^+ + W^-$, should be equal to $\frac{n(n+1)}{2}$, where n is the number of pairs of observations in the sample).

Case 2: Single set of observations

1. State the null hypothesis - the median value is equal to some value M .
2. Calculate the difference between each observation and the hypothesised median, $d_i = x_i - M$.
3. Apply Steps 3-5 as above.

Under the null hypothesis, we would expect the distribution of the differences to be approximately symmetric around zero and the the distribution of positives and negatives to be distributed at random among the ranks. Under this assumption, it is possible to

work out the exact probability of every possible outcome for W . To carry out the test, we therefore proceed as follows:

6. Choose $W = \min(W^-, W^+)$.
7. Use tables of critical values for the Wilcoxon signed rank sum test to find the probability of observing a value of W or more extreme. Most tables give both one-sided and two-sided p-values. If not, double the one-sided p-value to obtain the two-sided p-value. This is an exact test.

Normal approximation

If the number of observations/pairs is such that $\frac{n(n+1)}{2}$ is large enough (> 20), a normal approximation can be used with $\mu_W = \frac{n(n+1)}{4}$, $\sigma_W = \sqrt{\frac{n(n+1)(2n+1)}{24}}$

Dealing with ties:

There are two types of tied observations that may arise when using the Wilcoxon signed rank test:

- Observations in the sample may be exactly equal to M (i.e. 0 in the case of paired differences). Ignore such observations and adjust n accordingly.
- Two or more observations/differences may be equal. If so, average the ranks across the tied observations and reduce the variance by $\frac{t^3-t}{48}$ for each group of t tied ranks.

Example:

The table below shows the hours of relief provided by two analgesic drugs in 12 patients suffering from arthritis. Is there any evidence that one drug provides longer relief than the other?

Case	Drug A	Drug B	Case	Drug A	Drug B
1	2.0	3.5	7	14.9	16.7
2	3.6	5.7	8	6.6	6.0
3	2.6	2.9	9	2.3	3.8
4	2.6	2.4	10	2.0	4.0
5	7.3	9.9	11	6.8	9.1
6	3.4	3.3	12	8.5	20.9

Solution:

1. In this case our null hypothesis is that the median difference is zero.
2. Our actual differences (Drug B - Drug A) are:

$$+1.5, +2.1, +0.3, -0.2, +2.6, -0.1, +1.8, -0.6, +1.5, +2.0, +2.3, +12.4$$

Our actual median difference is 1.65 hours.

3. Ranking the differences and affixing a sign to each rank (steps 3 and 4 above):

Diff.	0.1	0.2	0.3	0.6	1.5	1.5	1.8	2.0	2.1	2.3	2.6	12.4
Rank	1	2	3	4	5.5	5.5	7	8	9	10	11	12
Sign	-	-	+	-	+	+	+	+	+	+	+	+

Calculating W^+ and W^- gives:

$$W^- = 1 + 2 + 4 = 7 \quad W^+ = 3 + 5.5 + 5.5 + 7 + 8 + 9 + 10 + 11 + 12 = 71$$

Therefore, we have $n = \frac{12 \times 13}{2} = 78$ and $W = \max(W^-, W^+) = 71$.

We can use a normal approximation in this case. We have one group of 2 tied ranks, so we must reduce the variance by $\frac{8-2}{48} = 0.125$. We get:

$$z = \frac{71 - \frac{12 \times 13}{4}}{\sqrt{\frac{12 \times 13 \times 25}{24} - 0.125}} = \frac{71 - 39}{\sqrt{162.5 - 0.125}} = 2.511$$

This gives a two-sided p-value of $p = 0.012$. There is strong evidence that Drug B provides more relief than Drug A.

3 Carrying out the Wilcoxon signed rank sum test in SPSS

- Choose **Analyze**
- Select **Nonparametric Tests**
- Select **2 Related Samples**
- Specify which two variables comprise your pairs of observation by clicking on them both then clicking on the arrow to put them under **Test Pair(s) List**.
- Under **Test Type** select **Wilcoxon**
- If you want exact probabilities (i.e. based on the binomial distribution), click on **Exact**, choose **Exact**, then **Continue**
- Click on **OK**

The output will look like this:

Ranks

	N	Mean Rank	Sum of Ranks
Drug B - Drug A Negative Ranks	3 ^a	2.33	7.00
Positive Ranks	9 ^b	7.89	71.00
Ties	0 ^c		
Total	12		

- a. DRUGB < DRUGA
- b. DRUGB > DRUGA
- c. DRUGA = DRUGB

Test Statistics^b

	DRUGB - DRUGA
z	-2.511 ^a
Asymp. Sig. (2-tailed)	0.012
Exact Sig. (2-tailed)	0.009
Exact Sig. (1-tailed)	0.004
Point probability	0.001

- a. Based on negative ranks
- b. Wilcoxon Signed Ranks Test